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LINEAR CONTROL SYSTEM OPTIMIZATION USING A MODEL-BASED INDEX OF PERFORMANCE.

UNITED STATES
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Research Report

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ABSTRACT

This paper deals with a method of optimizing the free coefficients in the characteristic equation of a linear feedback control system. The optimization is carried out by minimizing an index of performance associated with the system's response to a given test disturbance.

The index of performance is the integral of a quadratic function of the system state variables. The structure of the index rests upon a logical interpretation of the regulator nature of the control problem. The index for an $n^{\underline{th}}$ order system contains n weighting factors whose values are determined from an $n^{\underline{th}}$ order model system. This determination is such that the optimization of a completely free system will yield the model system. A system with fewer than n degrees of freedom in the state variable feedbacks may be optimized with respect to the free feedback coefficients, yielding a system whose dynamic response to a given disturbance is, for this optimization scheme, a best approximation to that of the model.

Examples are presented for illustration of the salient features of the method. It is also shown by example that systems with closed-loop zeros may be optimized by this method.

TABLE OF CONTENTS

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LIST OF ILLUSTRATIONS

CHAPTER I

INTRODUCTION

The task of designing a feedback control system normally begins with a mathematical modeling of the open-loop system, process, or plant that is to be controlled. This modeling permits an analytical treatment whereby the response of the closed-loop system is determined for a given test input. This response is very likely, at first, to be unsatisfactory with respect to a given set of specifications; compensation of the control system is then indicated.

Compensation is to some extent a matter of judgement; i.e., the choice of a compensation scheme depends upon such factors as cost, physical limitations of equipment, availability of certain items of information as readable signals, etc., all tempered by the experience of the designer.

It is now assumed that a compensation scheme has been chosen by the designer, and that the resulting compensated closed-loop transfer function has no zeros* and may be written

$$\text{CLTF} = \frac{k}{s^n + a_n s^{n-1} + a_{n-1} s^{n-2} + \ldots + a_1} \qquad (1)$$

where the a's and k are functions of the various plant and compensator parameters. The signal-flow graph of Figure 1 represents the transfer function of equation (1).

* This restriction will later be relaxed.

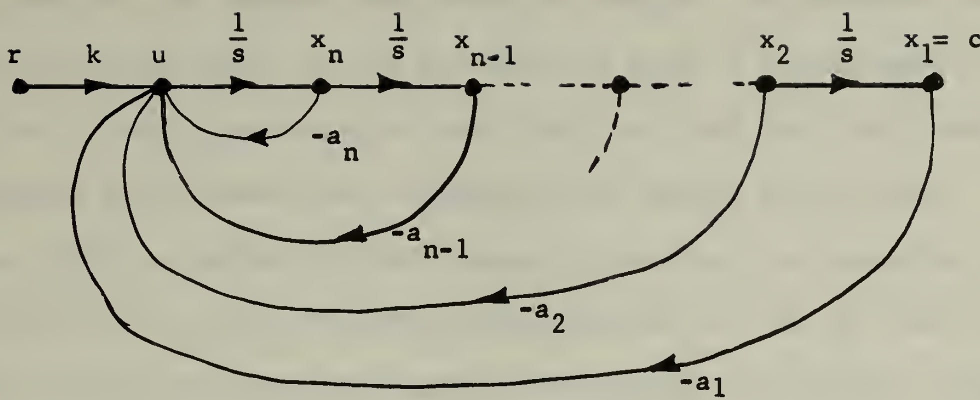

Fig. 1 Signal-flow graph for the transfer function of equation (1)

It is at this point that the designer must translate the given time and/or frequency domain specifications into desired root values for the characteristic polynomial, or equivalently, the poles of the closed-loop transfer function. If all n coefficients in the characteristic equation were freely variable, the n poles could conceivably be placed at any pre-selected combination of positions. This, of course, is rarely the case, since some of the a_n's contain or represent fixed plant parameters. As a rule, one free parameter is required for each distinct pole that is to be placed in a pre-selected position, or conversely, as many poles may be arbitrarily placed as there are free coefficients in the characteristic equation.** Magnitude constraints and other physical limitations may further restrict the placement of poles to certain regions in the s-plane.

** A complex conjugate pair may have one or the other of its polar coordinates (zeta, omega) fixed by one free coefficient.

On the ability to pre-select some of the poles of the system rests the basis of the dominant mode method of design. The designer uses as many free coefficients as are available to place a corresponding number of the n poles in pre-selected positions, with the hope that these poles so placed will dominate the response of the system to any input. Whether or not they are indeed dominant depends upon where the remaining poles are found to lie, how they make their presence felt in the time or frequency domain, and upon the degree of extra-pole interference judged to be tolerable by the designer. It is to be noted that the presence of closed-loop zeros further complicates the dominance picture, very possibly destroying the hoped-for dominance of a carefully placed set of poles.

The dominant mode method places emphasis on a limited number of the system's closed-loop poles in the hope that they will dominate the system dynamics. It is shown herein that more attention might profitably be directed toward placing all the system's poles in such a way that no group of poles is necessarily dominant, while all the poles collectively give system dynamics closely emulating those of the specifications. Such a procedure would start by establishing an ideal or model system of the form of Figure 1. This model system would be of the same order as the actual system, and the a_n coefficients would be selected on the basis of the given specifications and their translation to suitable pole values. A similar diagram, drawn for the actual system, would show some of the feedback coefficients as functions of fixed system parameters, the remainder as variable functions of the

free plant or compensator parameters. The presence of fixed parameters in the actual system effectively constrains the eventual design to certain definite regions in the s-plane. The design proceeds by selecting values of the variable coefficients in the constrained, actual system in such a way that it closely approximates the ideal model in its dynamic behavior in response to a given disturbance. This selection is to be made through the mechanism of a cost function, or index of performance, the fundamental structure of which is based upon intuitive judgement and some mathematical requirements. The index is so fashioned that it contains n undetermined weighting factors, where n is the order of the system being considered. To derive an algorithm for the determination of these factors for all systems of order n, an optimization, or minimization of the index is first carried out in such a way that the n weighting factors are forced to be functions of the n model feedback coefficients, the a_n's.

The actual n^{th} order system is once again considered. The index becomes, at this point, a function of some fixed system parameters, some variable parameters whose values are to be optimized, the n weighting factors whose values are determined by the designer's choice of a model system, and the n state variables as they respond to a given disturbance. If n parameters in the actual system were free, minimization of the index would drive those parameters to the model values, giving the desirable result of identical model and actual systems. If m $(m < n)$ of the actual coefficients were fixed, minimization of the index with respect to the remaining n-m free coefficients would

yield a system optimum with respect to an index of performance not entirely arbitrary, but based in good measure upon the pole locations of an ideal model system. In this sense, then, the system so designed is a "best" approximation to the model.

In the literature dealing with the optimum design of linear feedback control systems, much reference is made to arbitarily selected indices of performance, many of which will either not allow the selection of finite values for certain parameters, or will not allow optimization with respect to more than one system parameter; e.g., integral square error. Furthermore, optimization with respect to such indices often seems too much like an end in itself, rather than a means to the end of procuring system dynamics that will meet given time and frequency domain specifications.

Some attempts have been made to overcome the deficiencies mentioned above. Rekasius[5] has proposed an index based on a model differential equation of order less than that of the actual system being designed. This approach somewhat restricts the flexibility available to the designer in constructing a satisfactory model. The present method also allows optimization with respect to an index of performance; a single measure, to be sure, but one which incorporates such time and frequency domain specifications as bandwidth, rise time, and peak overshoot through the correlation of model dynamic response to model pole location and the dependence of the index's weighting factors on these pole locations. Furthermore, the model and actual systems are necessarily of the same order.

The chapters that follow will show the main steps in the selection of the model-based index of performance and the derivation of the algorithm by which the n weighting factors are determined. The algebraic details of the derivation are given in the appendix. The index is generalized for the n^{th} order system with no zeros in the closed-loop transfer function. Included are several examples of the use of the index in system design, for demonstration of the salient features of the method. It is also shown by example that the index may be used in the design of systems including zeros.

CHAPTER II

DERIVATION OF THE INDEX OF PERFORMANCE

The index of performance, or cost function[1], that is to link the actual and model system behavior will by choice be a quadratic integral functional of the form

$$J = \int_0^\infty (\underline{x}^T Q_1 \underline{x} + \underline{u}^T Q_2 \underline{u})\, dt \tag{2}$$

where $\underline{x}$ and $\underline{u}$ are as shown in Figure 1. A quadratic form is chosen simply because it admits of some mathematical treatment that would be very difficult if not impossible had another form been chosen. The differential equation of the system of Figure 1 may be written in vector matrix form as

$$\dot{\underline{x}} = G\underline{x} + \underline{u} \tag{3}$$

where G is given by

$$G = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & \ddots & 1 \\ & & & & 0 \end{bmatrix} \tag{4}$$

from

$$\dot{x}_m = x_{m+1} \quad ; \quad m < n \tag{5a}$$

and

$$\dot{x}_n = u \tag{5b}$$

For convenience, system response will be initiated by initial conditions

rather than by external inputs, resulting in a u matrix of the form

$$\underline{u} = A\underline{x} = \begin{bmatrix} & \bigcirc & \\ -a_1 & -a_2 \; \cdots\cdots & -a_n \end{bmatrix} \underline{x} \tag{6}$$

Equation (3) may now be written

$$\begin{aligned} \dot{\underline{x}} &= G\underline{x} + A\underline{x} \\ &\triangleq F\underline{x} \end{aligned} \tag{7}$$

The transpose of equation (6) may be substituted into equation (2) to give

$$\begin{aligned} J &= \int_0^\infty \left[\underline{x}^T (Q_1 + A^T Q_2 A)\, \underline{x}\right] dt \\ &\triangleq \int_0^\infty \underline{x}^T Q \underline{x}\, dt \end{aligned} \tag{8}$$

Except for the form of the Q_1 and Q_2 matrices, equations (2) through (8) completely describe the model system and its associated cost function.

The evaluation of J for an arbitrary set of initial conditions is to be carried out using vector matrix methods. Consider a Liapunov function of the system;

$$V(x) \triangleq \underline{x}^T P \underline{x} \tag{9}$$

Where P is a positive-definite, symmetric matrix. V(x) possesses the following properties[2], provided the system is stable:

$$\text{a)}\quad V(x) > 0, \quad x \neq 0$$
$$= 0, \quad x = 0$$
$$\text{b)}\quad \dot{V}(x) = \frac{dV(x)}{dt} < 0, \; x > 0 \qquad (10)$$
$$= 0, \; x = 0$$
$$\text{c)}\quad V(x) \to \infty \text{ as } |x| \to \infty$$
$$\text{d)}\quad V(x) \to 0 \text{ as } t \to \infty$$

Differentiation of equation (9) with respect to time yields

$$\dot{V}(x) = \dot{\underline{x}}^T P\underline{x} + \underline{x}^T P\dot{\underline{x}} \qquad (11)$$

Using the transpose of equation (7) in equation (11) gives

$$\dot{V}(x) = \underline{x}^T (F^T P + PF)\, \underline{x} \qquad (12)$$

To this point, the P matrix and the V(x) function associated with it are not defined. This definition is established by

$$\dot{V}(x) = \underline{x}^T (F^T P + PF)\, \underline{x}$$
$$\triangleq \underline{x}^T (-Q)\, \underline{x} \qquad (13)$$

Equation (13) is now integrated as follows:

$$\int_0^\infty \dot{V}(x)dt = - \int_0^\infty \underline{x}^T Q\underline{x}\; dt \qquad (14)$$

The right-hand side of the preceeding equation is immediately recognized as the negative of the cost function, equation (8). Evaluating the left-hand side of (14) and substituting limits yields

$$J = - V\left[x(\infty)\right] + V\left[x(0)\right] \qquad (15)$$

For a stable system, $x(\infty) \to 0$, and from the properties given in equations

(10), V [0] = 0. Therefore

$$J = V\left[x(0)\right] = \underline{x}(0)^T P\, \underline{x}(0) \tag{16}$$

Equation (16) reveals that the cost incurred by the system in response to a set of initial conditions is a function only of those initial conditions and of the P matrix. The P matrix in turn is a function of the system's F matrix and of the Q matrix, as given in equation (13). There remains, then, the task of fashioning the Q matrix to fulfill the mission set forth in the introduction and above. The following requirements, some mathematical and others intuitively logical, are to be considered in the selection of the Q_1 and Q_2 matrices:

a) Q must be a square, nxn, symmetric matrix.

b) Q is to be such that all states are included in J. This is a logical consequence of the regulator nature of the problem. Since all states are to be driven to zero, they should all contribute in some measure to the cost, the subsequent minimization of which yields an optimum transition to the origin of the state space.

c) Q_1 and Q_2 are to be such that together they introduce n weighting factors, whose values will ultimately depend upon the n feedback coefficients of the model system, or, equivalently, upon the model poles.

d) Q is to be such that J includes the "control effort" variable $\underline{u}$. This will act as a constraint to prevent results calling for infinite system parameters. It also insures that

the optimum control is a linear function of the system states[4].

In the light of the requirements stated above, the following configurations are suggested for the Q_1 and Q_2 matrices:

$$Q_1 = \begin{bmatrix} 1 & & & & \\ & \alpha_2 & & & 0 \\ & & \alpha_3 & & \\ & 0 & & \ddots & \\ & & & & \alpha_n \end{bmatrix}$$

and (17)

$$Q_2 = \begin{bmatrix} 1 & & & & \\ & 1 & & & 0 \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix} \lambda; \quad \lambda \text{ is a scalar}$$

where α_2 through α_n and λ constitute the n weighting factors. These combine by equation (8) to give

$$Q = \begin{bmatrix} 1+\lambda a_1^2 & \lambda a_1 a_2 & \cdots & \lambda a_1 a_n \\ \lambda a_2 a_1 & \alpha_2 + \lambda a_2^2 & \cdots & \lambda a_2 a_n \\ \vdots & & \ddots & \vdots \\ \lambda a_n a_1 & \cdots & & \alpha_n + \lambda a_n^2 \end{bmatrix} \tag{18}$$

The P matrix may now be evaluated literally, from equation (13). This is done for the second and third-order systems in the appendix, with results included for the fourth and fifth-order models as well. The

second-order results are presented here for illustration.

$$p_{11} = -q_{12} + \frac{a_2}{2a_1} q_{11} + \frac{a_1}{2a_2}\left(q_{22} + \frac{q_{11}}{a_1}\right)$$

$$p_{12} = p_{21} = \frac{q_{11}}{2a_1} \tag{19}$$

$$p_{22} = \frac{1}{2a_2}\left(q_{22} + \frac{q_{11}}{a_1}\right)$$

With the appropriate q_{ij}'s substituted in equations (19),

$$p_{11} = \frac{a_2}{2a_1} + \frac{a_1\alpha_2}{2a_2} + \frac{1}{2a_2} + \frac{\lambda a_1^2}{2a_2}$$

$$p_{12} = p_{21} = \frac{1}{2a_1} + \frac{\lambda a_1}{2} \tag{20}$$

$$p_{22} = \frac{\alpha_2}{2a_2} + \frac{\lambda a_2}{2} + \frac{1}{2a_1a_2} + \frac{\lambda a_1}{2a_2}$$

Equation (16) is now written in scalar form to show the relationship between J, the initial conditions, and the elements of the P matrix.

$$J = x_1^2(0)\ p_{11} + 2p_{12}\ x_1(0)\ x_2(0) + x_2^2(0)\ p_{22} \tag{21}$$

What follows is the key point of the entire derivation. <u>The values of α_2 and λ must be found so that, for the unconstrained system, the minimization of J with respect to a_1 and a_2 gives resulting optimum values of a_1 and a_2 that are, independently of the initial conditions, equal to the model values</u>; i.e., it is necessary to find the elements α_2 and λ as functions of the desired coefficients a_1 and a_2 so that the adjustment of the actual coefficients to the desired values will simultaneously minimize the J of equation (21) for any set of initial

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conditions.

The measures indicated above are carried out by first taking the gradient of J with respect to a_1 and a_2.

$$
\begin{aligned}
\frac{\partial J}{\partial a_1} &= x_1^2(0)\,\frac{\partial p_{11}}{\partial a_1} + 2x_1(0)\,x_2(0)\,\frac{\partial p_{12}}{\partial a_1} + x_2^2(0)\,\frac{\partial p_{22}}{\partial a_1} = 0 \\
\frac{\partial J}{\partial a_2} &= x_1^2(0)\,\frac{\partial p_{11}}{\partial a_2} + 2x_1(0)\,x_2(0)\,\frac{\partial p_{12}}{\partial a_2} + x_2^2(0)\,\frac{\partial p_{22}}{\partial a_2} = 0
\end{aligned}
\qquad (22)
$$

The initial conditions will be assumed to be generally non-zero and completely independent. The only possible solution to equations (22) must then be of the form

$$
\frac{\partial p_{ij}}{\partial a_i} = 0; \quad i,\ j = 1,\ldots.n \qquad (23)
$$

Substitution of equations (20) into equations (23) yields

$$
\lambda = \frac{1}{a_1^2} \qquad (24)
$$

and

$$
\alpha_2 = \frac{a_2^2 - 2a_1}{a_1^2} \qquad (25)
$$

Similar developments have been carried out for the third and fourth-order systems, some details of which are included in the appendix. At this point, a recursion formula for the structure of the α's in an n^{th} order model was postulated on the basis of second, third, and fourth-order model calculations. From this recursion formula, equation (26), a prediction was made for the fifth-order

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case, and was proved correct by extremely lengthy but straightforward algebra. This successful prediction of the fifth-order α's has been accepted as sufficient proof of the validity of the recursion formula.

$$\alpha_{i,n} = \frac{1}{a_1^2} \left[a_i^2 + 2 \sum_{k=1}^{n-1} (-1)^k a_{i-k} \cdot a_{i+k} \right] \tag{26}$$

and

$$\lambda = \frac{1}{a_1^2}$$

where n is the order of the model, and $1 \leq i \leq n$. By definition,

$$\begin{aligned} a_o &= 0 \\ a_{n+1} &= 1 \\ a_{(n+1+\ell)} &= 0 \; ; \; \ell > 0 \end{aligned} \tag{27}$$

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CHAPTER III

ILLUSTRATIVE EXAMPLES

EXAMPLE I

Consider the second-order feedback control system of Figure 2. The amplifier gain k is the only variable parameter, and is to be selected so that the system's transient response to a given disturbance is a best approximation to the model, or desired response. The model response has been chosen by the designer as the response characterized

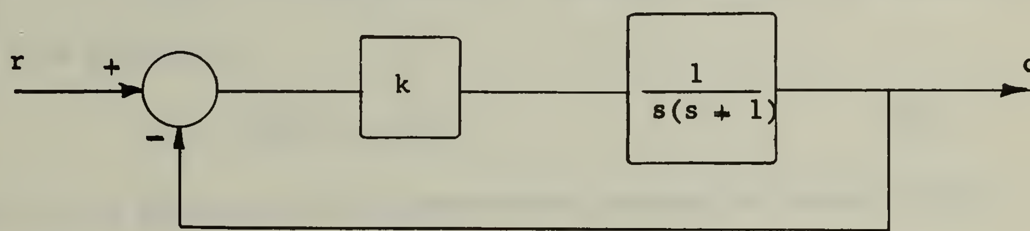


Fig. 2 Feedback control system of Example I

by closed-loop roots of damping ratio $\zeta = 0.7$ and natural frequency $\omega_n = 2.0$ radians per second. An inspection of the system root locus for $0 \leq k \leq \infty$ makes it immediately apparent that the ideal root location is unattainable. One may therefore turn to minimization of a performance index, as suggested in the previous pages, as a means of selecting k.

The system of Figure 2 is for present purposes best represented by the state variable signal-flow graph of Figure 3(a). The model system of Figure 3(b) reflects the desired but unattainable characteristic equation coefficients for the specified root locations.

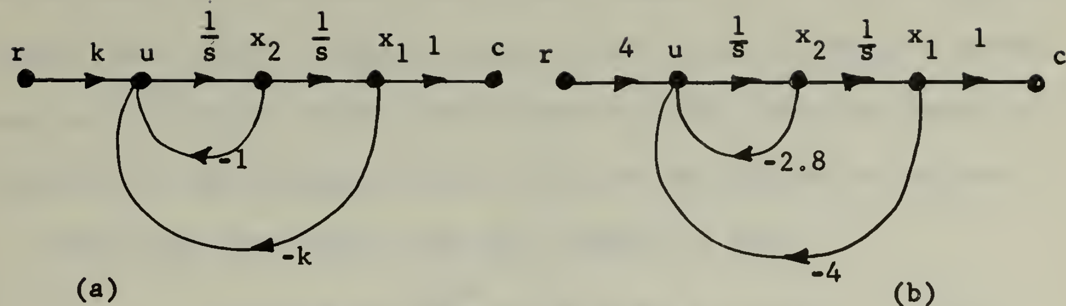


Fig. 3 Signal-flow graphs, Example I.
(a) actual, (b) model.
From characteristic equation $s^2 + a_2 s + a_1 = 0$

For convenience, let the system be disturbed by the initial conditions $x_1(0) = 1.0$, $x_2(0) = 0$, and $r(t) = 0$. The cost function of equation (21) then reduces to

$$J = x_1^2(0)\ p_{11} = p_{11} \tag{28}$$

where p_{11} is given, for this second-order system, by equation (19). The Q matrix is now to be evaluated. Substituting the model coefficients, $a_1 = 4.0$ and $a_2 = 2.8$ into equations (24) and (25) gives

$$\alpha_2 = -0.01$$

and (29)

$$\lambda = \frac{1}{16}$$

These values are then substituted in equation (18), yielding

$$Q = \begin{bmatrix} \left(1 + \frac{k^2}{16}\right) & \frac{k}{16} \\ \frac{k}{16} & \left(-0.01 + \frac{1}{16}\right) \end{bmatrix} \tag{30}$$

where $a_1 = k$ and $a_2 = 1.0$ are the actual system coefficients. It is

to be noted that the model coefficients appear in Q only through the quantities α_2 and λ. Whenever else in the Q matrix the a_n's appear, they are the actual system coefficients, either fixed and known, or functions of the parameters that are to be determined as optimum.

Equations (19), (28), and (30) combine to give

$$J = -0.005k + \frac{k^2}{32} + \frac{1}{2k} + \frac{1}{2} \tag{31}$$

Minimizing J with respect to k gives

$$\frac{\partial J}{\partial k} = -0.005 - \frac{1}{2k^2} + \frac{k}{16} = 0 \tag{32}$$

Equation (32) has the real solution

$$k = 2.03 \tag{33}$$

Figure 4 shows the system root locus and compares the model and actual root locations.

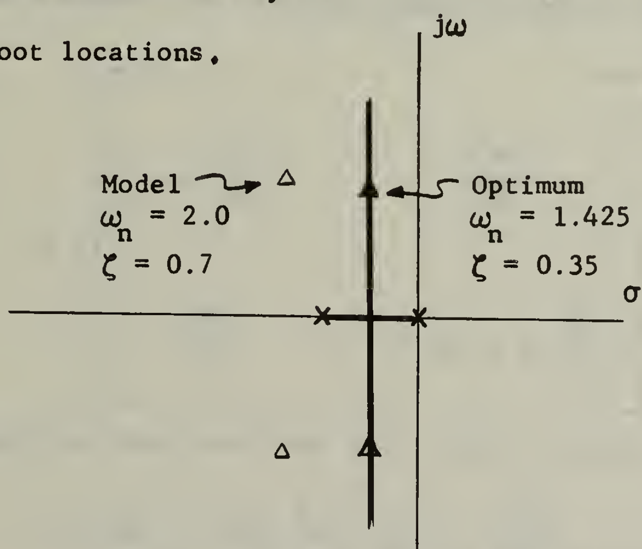


Fig. 4 Root locations for Example I

Figure 5 compares the model and actual transient responses, as simulated on the analog computer.

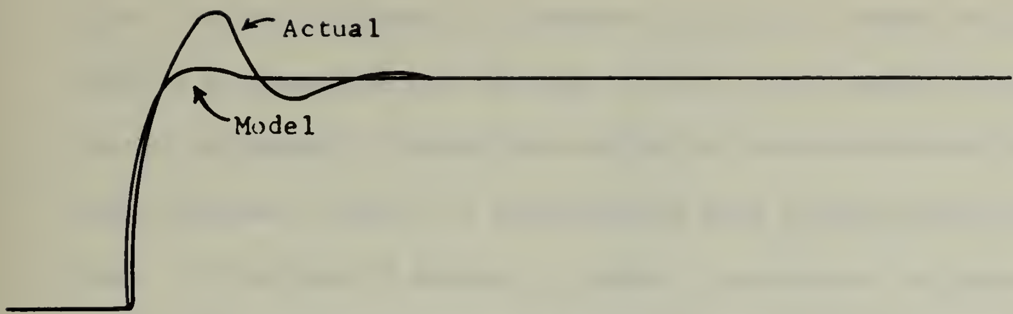


Fig. 5 Transient responses, Example I

Further insight into the method and means by which the optimum value of k was found may be obtained by investigation of the cost function associated with two free coefficients in Example I. If both coefficients in the characteristic equation are allowed to be free, the Q matrix becomes, for the same model values of $a_1 = 4.0$ and $a_2 = 2.8$,

$$Q = \begin{bmatrix} \left(1 + \frac{a_1^2}{16}\right) & \frac{a_1 a_2}{16} \\ \frac{a_1 a_2}{16} & \left(-0.01 + \frac{a_2^2}{16}\right) \end{bmatrix} \tag{34}$$

The cost function from equations (19), (28), and (34) is now

$$J = \frac{a_2}{2a_1} - 0.005 \frac{a_1}{a_2} + \frac{1}{2a_2} + \frac{a_1^2}{32a_2} \tag{35}$$

Figure 6 is a plot of the surface generated by equation (35). This surface is, for the initial conditions of the example, uniquely associated with the values of the model coefficients, $a_1 = 4.0$ and

$a_2 = 2.8$, and displays its minimum $J_{min\ min}$ at these very values. This is a consequence of the way in which the Q matrix was constructed, and of the manner in which the weighting factors were derived; complete freedom to vary all coefficients will always yield the model system. In the case of Example I, where a constraint is placed on one of the two coefficients in the form of $a_2 = 1.0$, it is not possible to drive to the absolute minimum on the cost surface. The constraint is equivalent to passing the plane $a_2 = 1.0$ through the cost surface. The optimum value of $a_1 = k$ is then that value which achieves a minimum J_{min} on the intersection of the plane and surface. From Figure 6, it is seen that this value is $k = 2.03$, in agreement with equation (33).

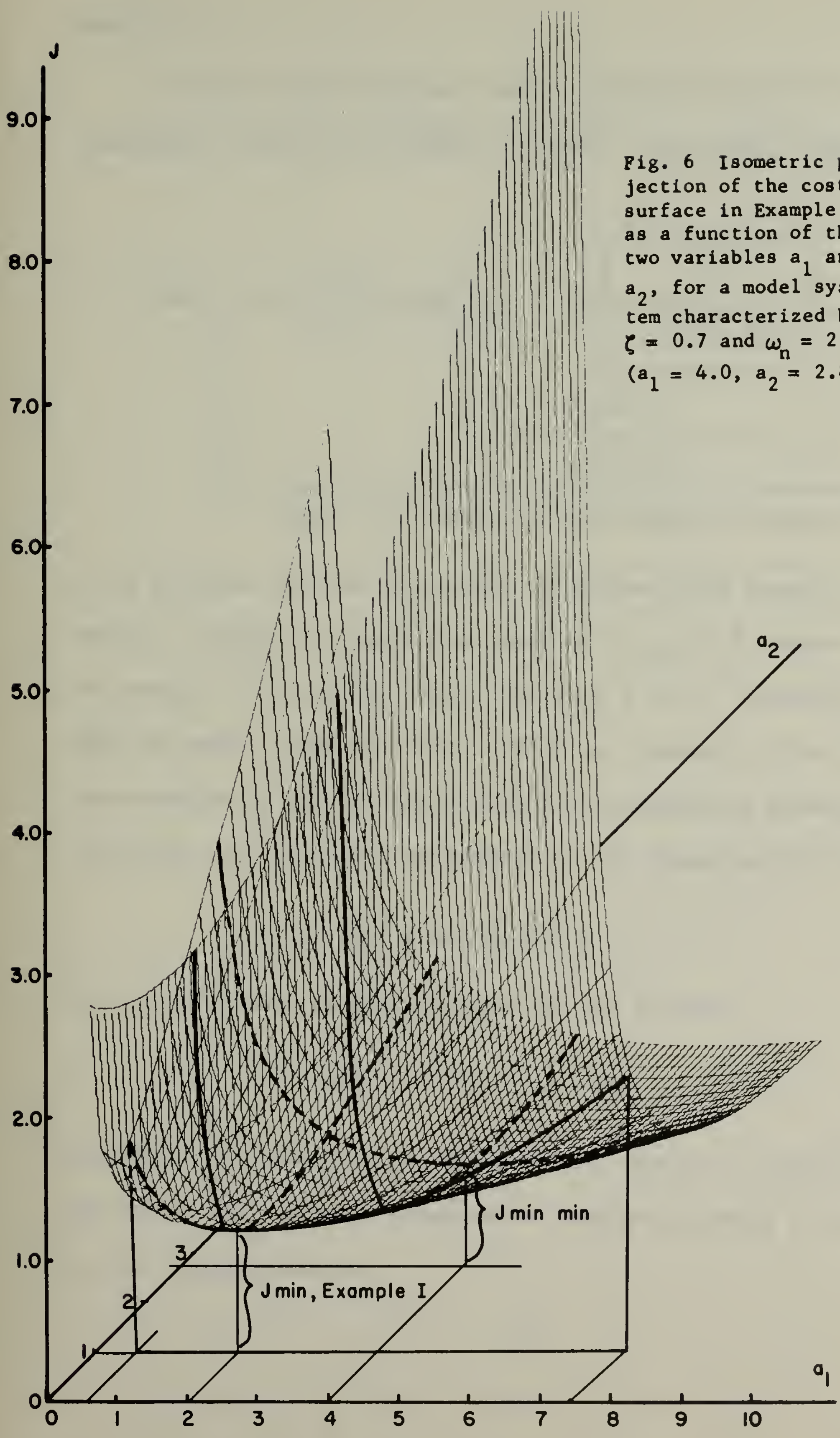
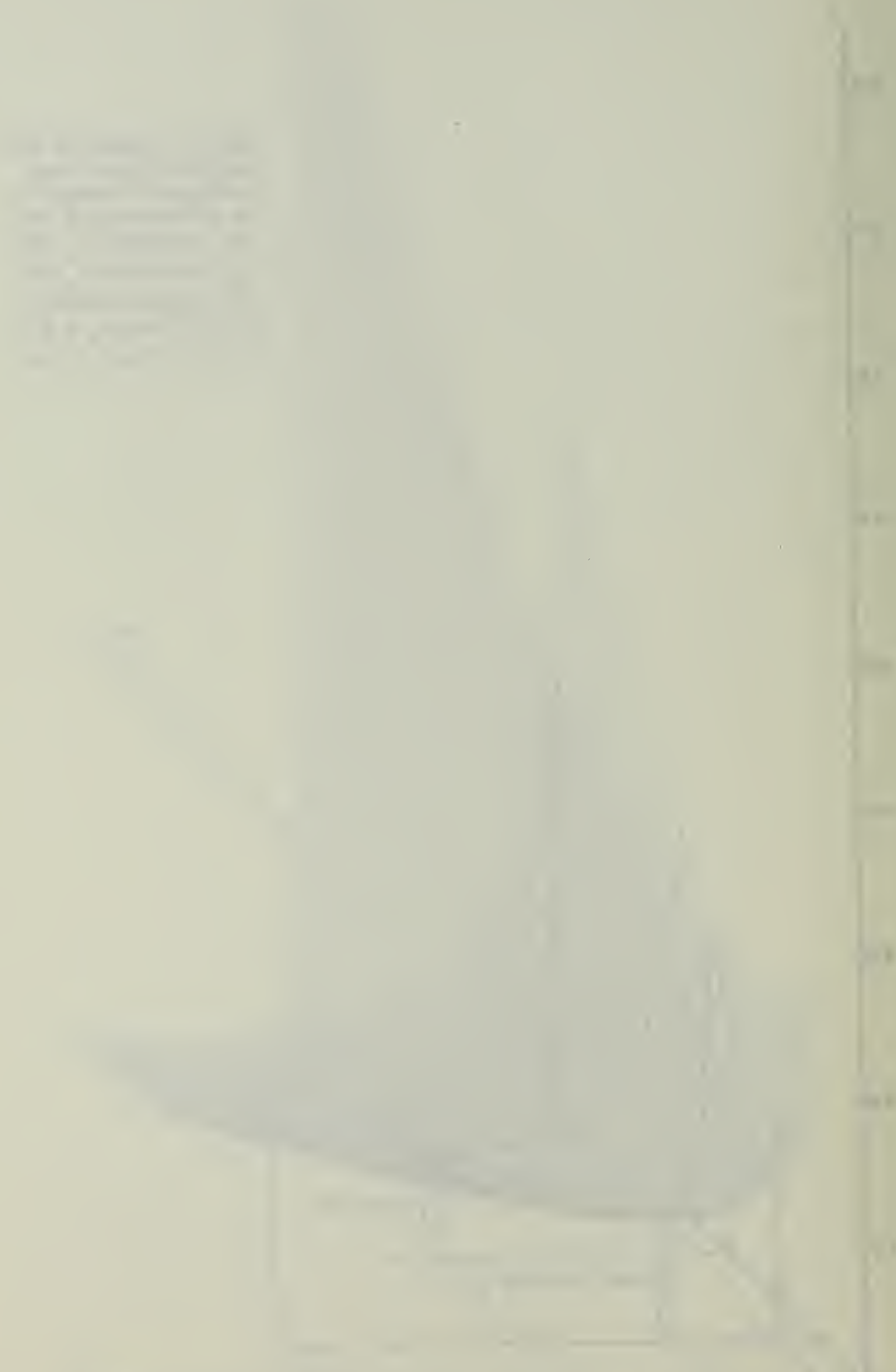


Fig. 6 Isometric projection of the cost surface in Example I, as a function of the two variables a_1 and a_2, for a model system characterized by $\zeta = 0.7$ and $\omega_n = 2.0$ ($a_1 = 4.0$, $a_2 = 2.8$)



EXAMPLE II

The third-order feedback control system of Figure 7 is to be compensated using rate feedback as shown. The gains h and k are

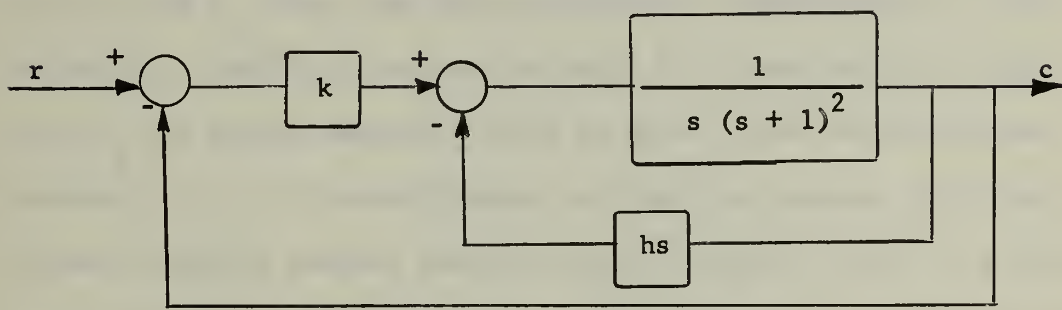


Fig. 7 Feedback control system of Example II

to be selected so that the system is optimum with respect to a model having a complex conjugate root pair with ω_n = 2.0 radians per second and ζ = 0.7, and the third root at s = -5.0. Inspection shows that the model poles have been placed in a region of the s plane made unattainable by the constraints of the compensation scheme chosen. The model root locations correspond to the characteristic equation

$$s^3 + 7.8s^2 + 18s + 20 = 0 \qquad (36)$$

The actual system is characterized by the equation

$$s^3 + 2s^2 + (h + 1)s + k = 0 \qquad (37)$$

Equation (37) shows that a_3 in the actual system is constrained to the value 2.0, and that there exists complete freedom to vary the a_1 and a_2 coefficients.

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To show the single cost surface associated with three degrees of coefficient freedom is, of course, graphically impossible. The constraint $a_3 = 2.0$, however, is effectively a plane which intersects the hypersurface in such a way as to yield a subsurface dependent upon a_1 and a_2 alone, and hence graphically representable. Equivalently, a family of surfaces giving J as a function of a_1 and a_2 with a_3 the family parameter could be drawn. The surface corresponding to $a_3 = 7.8$ would display the absolute minimum of all surfaces, while the surface drawn for $a_3 = 2.0$ would display a minimum greater than the absolute minimum, and would yield the optimum values of a_1 and a_2 and hence of h and k. Figure 8 is a plot of the subsurface for $a_3 = 2.0$, the equation for which is derived below.

From the recursion formula (26), for the third-order model with $a_3 = 7.8$, $a_2 = 18.0$, and $a_1 = 20.0$,

$$\alpha_2 = \frac{a_2^2 - 2a_1a_3}{s_1^2} = \frac{12}{400}$$

$$\alpha_3 = \frac{a_3^2 - 2a_2}{a_1^2} = \frac{24.8}{400} \tag{38}$$

$$\lambda = \frac{1}{400}$$

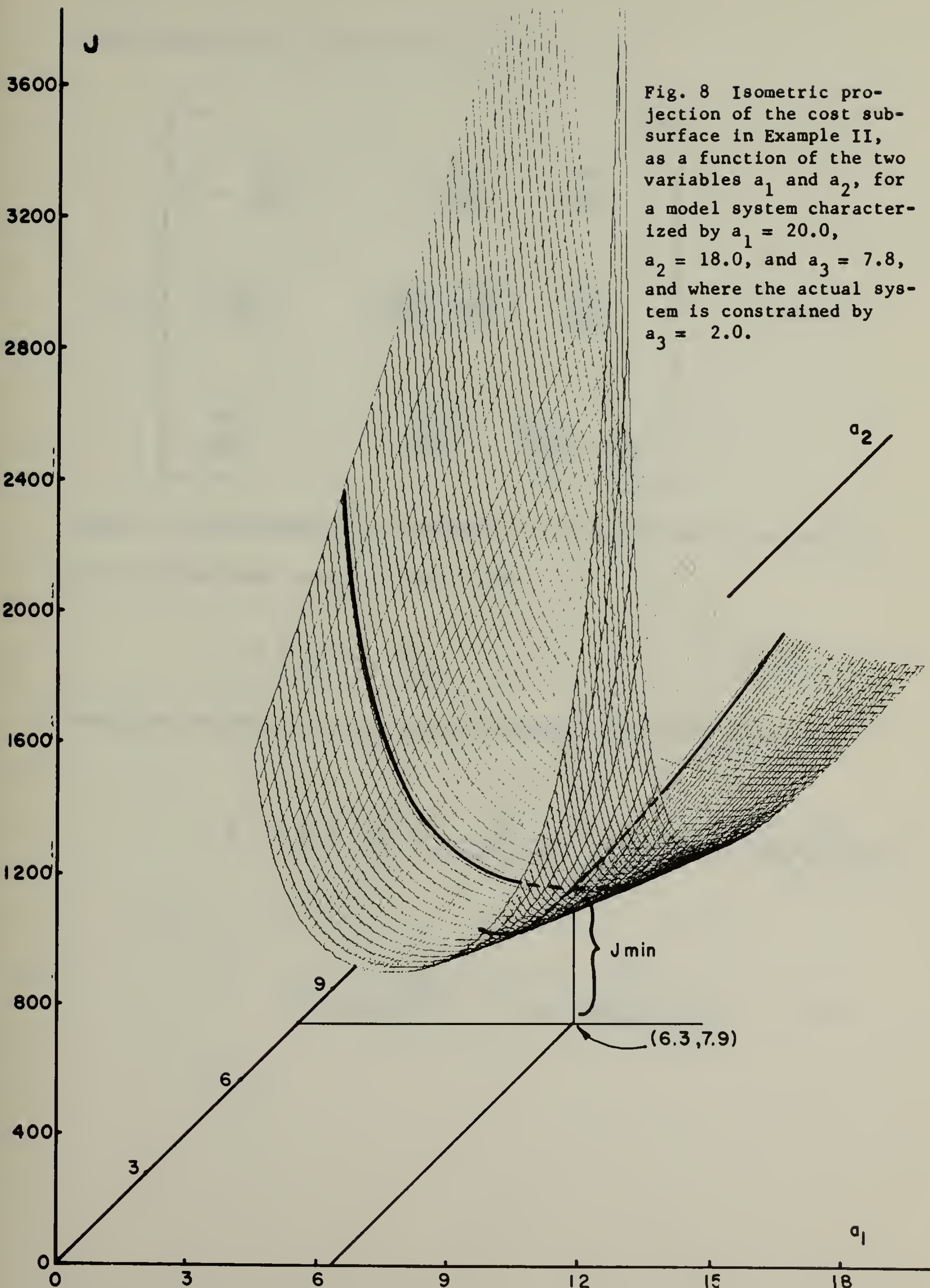

Fig. 8 Isometric projection of the cost subsurface in Example II, as a function of the two variables a_1 and a_2, for a model system characterized by $a_1 = 20.0$, $a_2 = 18.0$, and $a_3 = 7.8$, and where the actual system is constrained by $a_3 = 2.0$.

From equation (18), the Q matrix is

$$Q = \begin{bmatrix} (1 + \frac{a_1^2}{400}) & \frac{a_1 a_2}{400} & \frac{2a_1}{400} \\ \frac{a_1 a_2}{400} & (\frac{12}{400} + \frac{a_2^2}{400}) & \frac{2a_2}{400} \\ \frac{2a_1}{400} & \frac{2a_2}{400} & (\frac{24.8}{400} + \frac{4}{400}) \end{bmatrix} \qquad (39)$$

Again for convenience, the system will be disturbed by an initial unit displacement only, in which case

$$J = x_1^2(0)p_{11} = p_{11} \qquad (40)$$

From the appendix, p_{11} for the third-order system is given by

$$p_{11} = q_{11}\left[\frac{a_2}{2a_1} + \frac{a_3^2}{2(a_2 a_3 - a_1)}\right] - q_{12} + q_{22}\left[\frac{a_1 a_3}{2(a_2 a_3 - a_1)}\right]$$

$$- q_{13}\left[\frac{a_1 a_3}{a_2 a_3 - a_1}\right] + q_{33}\left[\frac{a_1^2}{2(a_2 a_3 - a_1)}\right] \qquad (41)$$

[illegible]

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$$

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[illegible]
$$

[illegible]

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From equations (39), (40), and (41), with $a_3 = 2.0$,

$$J = \frac{\left[\begin{array}{l} 800\ a_2^2 - 400\ a_1 a_2 + a_1^3 a_2 + 24 a_1^2 + \\ 24.8\ a_1^3 + 1600\ a_1 \end{array}\right]}{2a_1(2a_2 - a_1)} \tag{42}$$

By far the most reasonable method of finding a_1 and a_2 for J_{min} in equation (42) is by surface search techniques using a general-purpose digital computer. In this case the search was conducted by incrementing the parameters a_1 and a_2 about an initial point, testing for the maximum decrement in J, establishing a new initial point, and repeating the process until the minimum was located. J_{min} was found to lie at

$$\begin{aligned} a_1 &= 6.3 \\ a_2 &= 7.9 \end{aligned} \tag{43}$$

giving optimum gain settings, by equation (37), of

$$\begin{aligned} k &= 6.3 \\ h &= 6.9 \end{aligned} \tag{44}$$

Figure 9 compares the model and actual root locations, while Figure 10 compares the transient responses.

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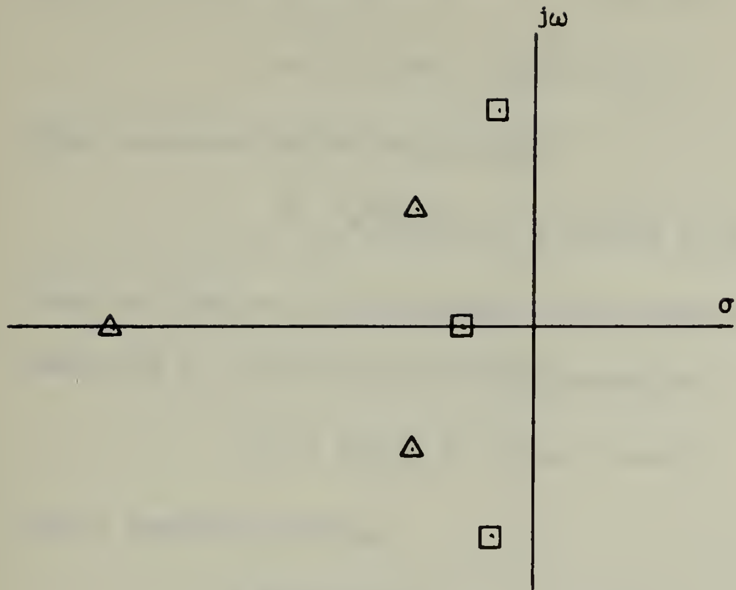


△ Model: $S = -5, -1.4 \pm j1.43$
□ Optimum: $S = -0.91, -0.54 \pm j2.56$

Fig. 9 Root locations for Example II

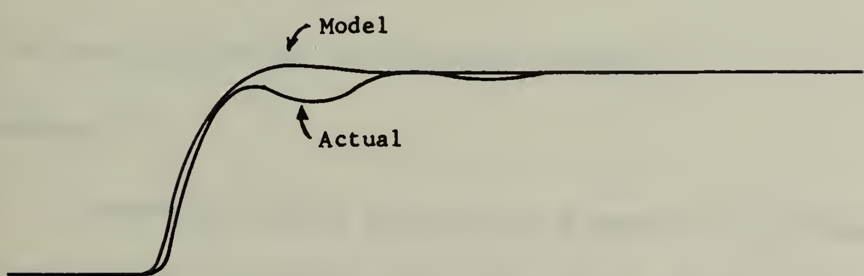


Fig. 10 Transient responses for Example II

It is of interest to note that the dominant mode method discussed in Chapter I, if applied to Example II, yields a peculiar result. Since two coefficients are independently free, it is reasonable to expect that a pair of complex conjugate roots may be exactly placed. No control is possible over the position of the third, real

root, p. If the conjugate pair is placed at $\zeta = 0.7$, $\omega_n = 2.0$,

$$(s^2 + 2.8s + 4)(s + p) = 0 \tag{45}$$

which upon multiplication becomes

$$s^3 + s^2(2.8 + p) + s(2.8p + 4) + 4p = 0 \tag{46}$$

Equation (46) is to be compared, with respect to coefficients of like powers of s, to the actual system equation

$$s^3 + 2s^2 + (h + 1)s + k = 0 \tag{47}$$

Such a comparison shows

$$\begin{aligned} p &= -0.8 \\ k &= -3.2 \\ h &= \ 0.76 \end{aligned} \tag{48}$$

The resulting system is obviously unstable.

EXAMPLE III

A feedback control system with a zero in its closed-loop transfer function does not fit the classification established by Figure 1. It will now be shown, however, that zeros may be accounted for by introduction of forward, non-looping paths from appropriate state variables in the signal-flow graph of Figure 1[3], and that the method of optimization employed in the previous examples may again be used without modification.

The closed-loop transfer function of the system in Figure 11 may be written

$$\frac{c}{r} = \frac{x_1}{r} \cdot \frac{c}{x_1}$$

$$= \frac{k}{s^3 + s^2(p + 1) + s(p + k) + kz} \cdot (s + z) \qquad (49)$$

which displays a closed-loop zero at $s = -z$. The signal-flow graph

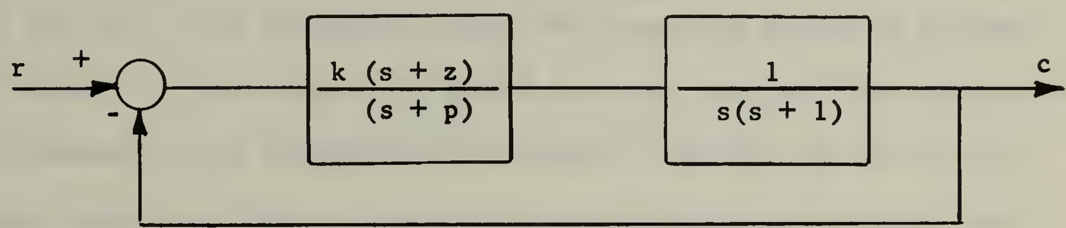


Fig. 11 Feedback control system of Example III

of equation (49) is shown in Figure 12. One notes that all the closed-

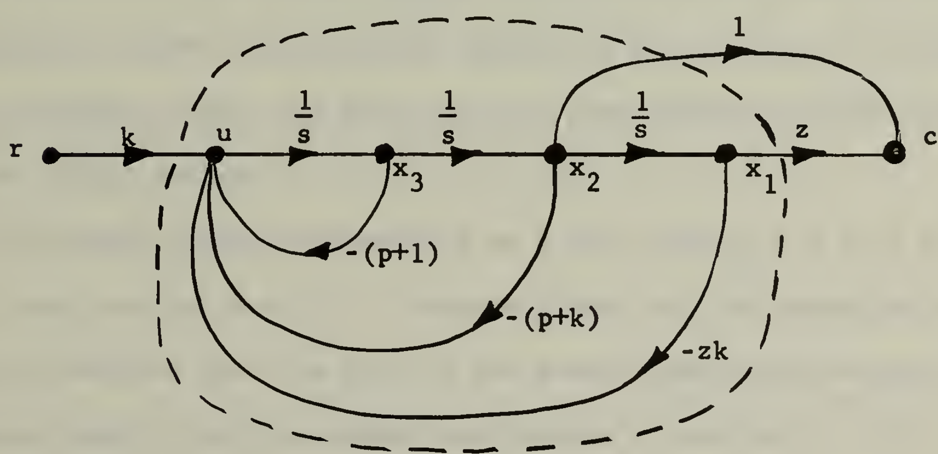
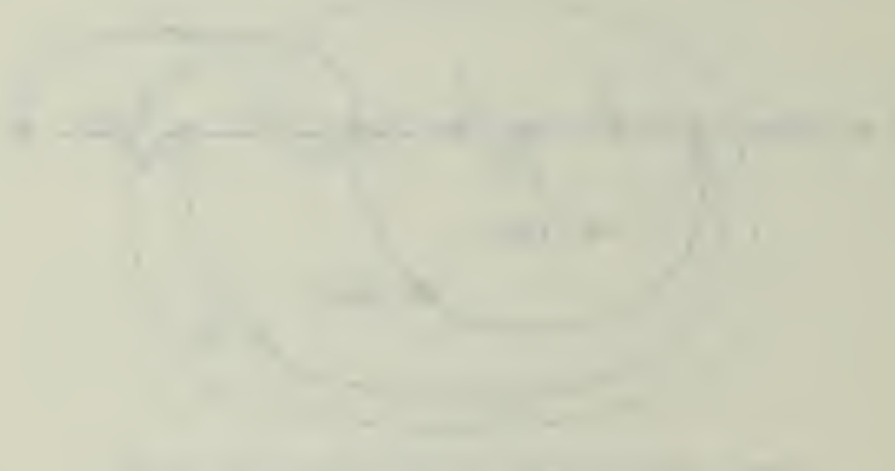


Fig. 12 Signal-flow graph of equation (49)

loop pole information is contained within the dotted line enclosing the feedback portions of the graph, and that this part of the graph is consistent with the form of Figure 1. From equation (49),



$$c = (s + z)x_1 \tag{50}$$

Equation (50) describes the feed-forward paths necessary to make up the output, c, and also accounts for the closed-loop zero.

The design objectives of this example may be stated by selecting model values for the three poles and one zero. The values of k, z, and p are then to be selected so that the resulting system is optimum with respect to this model. It should be noted that at first inspection, three parameters are independently variable, and that one could conceivable place the three closed-loop poles precisely where they were wanted and let the zero fall where it may. This, however, is not in keeping with the idea of driving a constrained actual system as close as possible toward a model whose respective pole and zero locations are considered ideal, and which has the same number of poles and zeros as the actual system.

The model poles are specified as a pair with $\omega_n = 2.0$, $\zeta = 0.7$, and a real pole at $s = -5.0$. The model zero will be placed at $s = -2.0$. One now observes that the zero of the actual system may be made to coincide exactly with the model zero simply by setting $z = 2.0$. This is an arbitrary choice but a necessary one, since the present method optimizes a response on the basis of pole location alone. There remains, then, the task of placing the three system poles so as to be optimum with respect to the model poles, using the two free parameters k and p. Because the zero has been dealt with and disposed of, only that part of Figure 12 lying within the dotted line is yet to be

determined. It is for this configuration that the optimization method of Chapters I and II has been derived.

With $z = 2.0$, the closed-loop transfer function becomes

$$\frac{c}{r} = \frac{k(s + 2)}{s^3 + s^2(p + 1) + s(p + k) + 2k} \tag{51}$$

Because the model pole locations of this example are identical to those of Example II, α_2, α_3, and λ are given by equations (38). Following the same steps as those taken between equations (39) and (42) of Example II, one finds

$$J = \frac{\left[\begin{array}{l} 400p^3 + 1600kp^2 + 448k^2p + 400p^2 + 1600kp \\ -352k^2 + 8k^3p + 230.4k^3 + 800k + 8k^4 \end{array}\right]}{1600k(p^2 + kp + p-k)} \tag{52}$$

Minimization of (52) with respect to k and p was carried out by surface search techniques and yielded the values

$$\begin{aligned} k &= 10.8 \\ p &= 8.0 \end{aligned} \tag{53}$$

The cost J could have been expressed in terms of the a_n coefficients rather than k and p, in which case

$$\begin{aligned} a_1 &= 2k \\ a_2 &= p + k \\ a_3 &= a_2 - a_1/2 + 1 \end{aligned} \tag{54}$$

The minimization would then have been carried out with respect to a_1 and a_2 rather than k and p. Algebraic solution for k and p would then

follow by equations (54).

Figure 13 compares model and actual pole and zero locations, while Figure 14 compares transient responses.

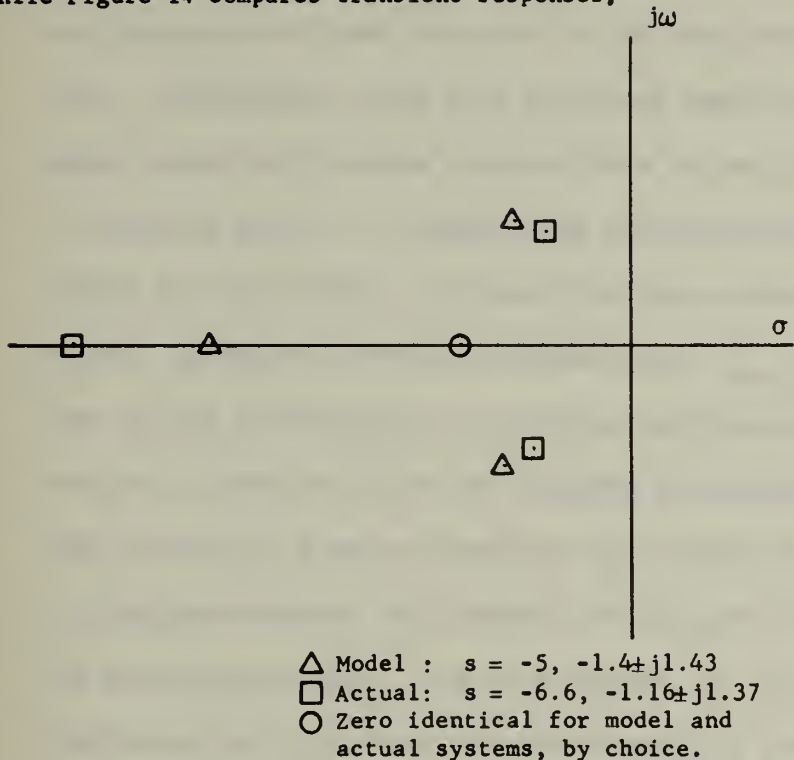


Fig. 13 Pole and zero locations for Example III

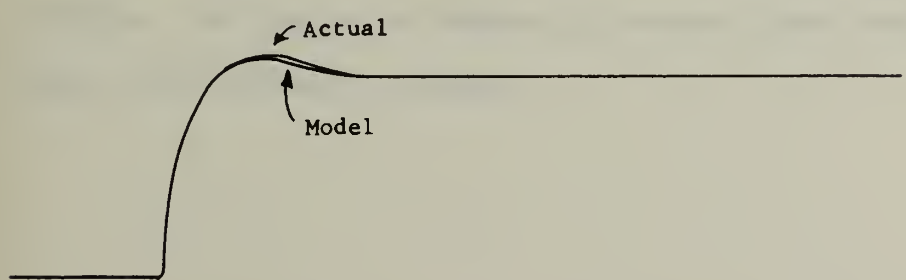


Fig. 14 Transient responses for Example III

It must be emphasized that the solution obtained above is not unique, since a different set of optimum parameters would have emerged had another value been selected in the free choice of the zero location. Furthermore, many such solutions would very likely show that among those many optimum systems there is one that is "most optimum" in terms of cost, J. Indeed, this solution would be the one first mentioned in this example, in which the three poles assume their model values, giving the absolute minimum cost, $J_{min\ min}$. Doubts might be cast on the acceptability of such an "optimum optimum", however, if the zero turned out to be far removed from the model zero, for the cost function J does not penalize the output of the system of Figure 12, and acknowledges the presence of the zero only insofar as it affects the closed-loop poles. For this reason, it is suggested that whenever free zeros are introduced, as for example by cascade compensation, these zeros be placed at or near the model zero values for a first solution. The results of Example III indicate that this approach can yield very satisfactory results.

CHAPTER IV

DISCUSSION AND CONCLUSION

A. STABILITY

The question of stability must be discussed in connection with any method of control system design. The method of the previous chapters will consistently yield a stable optimized system if a few straightforward guide-lines are followed in setting up any particular design problem.

The first requirement is that the model system be stable. This insures that there will exist a multi-dimensional cost surface, positive in a region surrounding the model parameter values, and exhibiting a minimum at these model values. The region of stability for the model cost surface has bounds determined only by the order of the system. The stable region for the surface of Figure 6, for example, is defined by $a_1 > 0$, $a_2 > 0$. For the total surface of Example II, of which Figure 8 is a subsurface, the stability region would be defined by $a_2a_3 - a_1 > 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, a fact which is made evident both by the Routh criterion and by equation (41).

The designer will normally choose a compensation scheme that provides a good measure of feedback from plant variables, simply because these variables contain information about the more basic but unavailable state variables. He may also choose to approximate the state variables by the use of various filter networks. In any event, it is more likely that the final choice of compensation will at least

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allow absolute stability. Any concern with stability is now ended, for the optimization process that follows stabilization cannot possibly render the system unstable. The fact that the system can be stable indicates that the real system's constraints on the coefficients of the characteristic equation are such that the cost subsurface generated by the constraints lies within the stable region of the model surface, and exhibits its minimum therein. Figure 8 shows just such a subsurface generated by a constraint $a_3 = 2.0$ in Example II. It should be noted that there is now a secondary stability region acknowledging the constraint, and defined by $2a_2 - a_1 > 0$, $a_1 > 0$, $a_2 > 0$. In searching the subsurface for its minimum and hence the optimum values for a_1 and a_2, care must be taken that the search begin at a point known to be within the stable region.

In summary, stability is handled by any of the conventional methods. By applying them, the designer insures that the compensated system can be stable. The stability region for the constrained surface (subsurface) is determined, and the search of the subsurface proceeds from a point known to be in the stable region.

B. COST SURFACE SELECTIVITY SENSITIVITY TO PLANT PARAMETER VARIATIONS

The term selectivity will be associated with the slope of the cost subsurface in the neighborhood of its minimum, or optimum point. High selectivity will indicate a relatively sharp minimum; low selectivity, a shallow minimum. Low selectivity in a cost function can be somewhat undesirable in certain contexts. Let it be assumed, for

example, that a system is being adaptively optimized with respect to a certain integral index of performance, and that the index, or cost, is being evaluated by actual integration of the appropriate functions of the system's state variables as they respond to a perturbation. It is further assumed that the state variables called for in the index are available and measurable. As soon as measurement enters the problem, measurement errors inevitably follow. It is easily seen that small errors in cost evaluation can bring about relatively large errors in the optimum parameter settings for the case in which the cost subsurface near the true minimum is inherently shallow.

The design method of this paper, however, does not in any way rely upon the measurement of dynamic variables, and is to that extent free of low selectivity shortcomings. Given that a cost subsurface has a true minimum, that minimum can be located with as much accuracy as computing machines will allow, be the minimum shallow or sharp.

Measurement errors of another kind do find their way into the present method via the identification of the plant's configuration and fixed parameters. The term sensitivity shall be employed to describe the relationship between a small change or error in a given fixed plant parameter and the corresponding displacement of the cost subsurface and its true minimum. A high sensitivity would indicate that relatively greater care would need be taken in determining the fixed constants of the plant being controlled.

Sensitivity may be dealt with quantitatively, as will now be shown using Example I for illustration. Let the open-loop plant

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pole in Figure 2 of Example I take on all values on the negative real axis; i.e., if the pole factor is designated as (s + p),

$$0 < p < \infty \tag{55}$$

Assume that the optimum value of k has been determined for enough values of p to establish the hypothetical curve of Figure 15. The

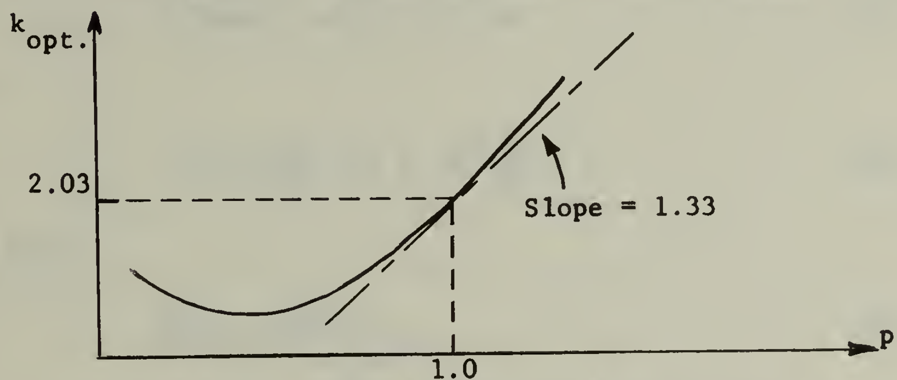


Fig. 15 Hypothetical curve of optimum gain k as a function of plant pole p for the system of Figure 2

sensitivity of the optimized parameter k to changes in the plant pole p is defined as

$$S_{kp} = \frac{dk}{dp} \tag{56}$$

of simply the slope of the k versus p curve.

The cost function of equation (31) may be rewritten to include the plant pole literally, as follows:

$$J = \frac{p}{2k} - 0.005\,\frac{k}{p} + \frac{1}{2p} + \frac{k^2}{32p} \tag{57}$$

Optimization is carried out with respect to k, giving

$$\frac{\partial J}{\partial k} = -\frac{0.005}{p} - \frac{p}{2k^2} + \frac{k}{16p} = 0 \tag{58}$$

Equation (58) may be solved to give the optimum k for any plant pole p, for the model system selected in Example I, $\omega_n = 2.0$, $\zeta = 0.7$. It may also be differentiated with respect to p to yield the sensitivity of the optimum gain to changes in the plant pole p. From equation (58),

$$-0.08k^2 - 8p^2 + k^3 = 0 \tag{59}$$

and

$$-0.16k \frac{dk}{dp} - 16p + 3k^2 \frac{dk}{dp} = 0 \tag{60}$$

from which

$$\frac{dk}{dp} = \frac{16p}{3k^2 - 0.16k} \tag{61}$$

Substituting the values p = 1.0 and k = 2.03 from Example I, gives

$$S_{kp} = \frac{dk}{dp} = 1.33 \tag{62}$$

Thus there has been added to the coordinates of a single point on the curve of Figure 15 the slope at which the curve passes through that point. Although no further information about the curve has been deduced, this figure for sensitivity gives assurance that small changes in the plant pole about the value p = 1.0 would not call for drastic changes in the optimum forward gain setting, and that no violent departure from optimum response will be observed if the gain setting is left at a single constant value.

It is believed that the sensitivity problem as described above presents a potentially fruitful area for further investigation.

C. POTENTIAL APPLICATION IN ADAPTIVE SYSTEMS

The examples in Chapter III deal with linear, time-invariant plants and the selection of compensator parameters to optimize performance relative to a model. The design is carried out but once, and remains valid as long as the plant remains unchanged. Should the plant constants change for any reason, the design becomes invalid in the strict sense of optimality with respect to the model. The extent to which plant parameter variations may be troublesome is in part a function of the sensitivity associated with each parameter. The effect of the variations can be counteracted by effectively redesigning the system constantly, or as often as new information about the changing plant parameter is made available. Thus the system would be adapting to changing conditions by continuous optimization.

Figure 16 shows a suggested functional block diagram of such an

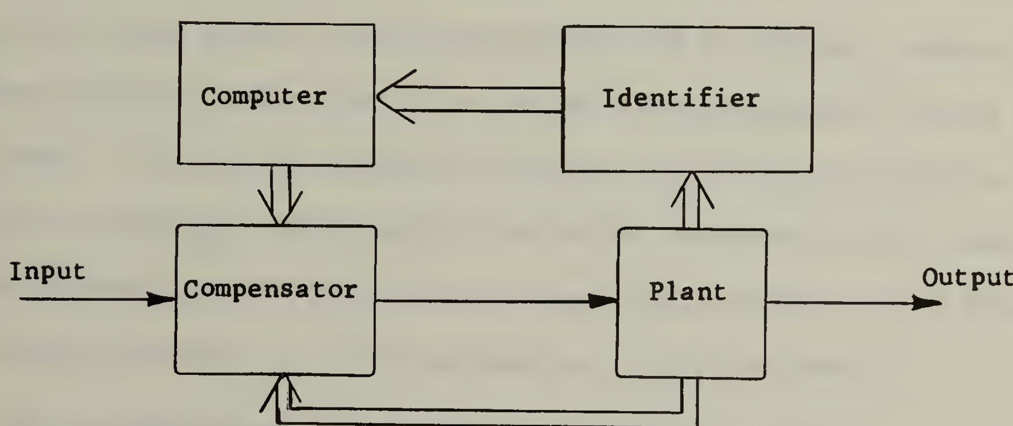


Fig. 16 An adaptive system using a digital computer for surface search and optimization.

adaptive system. It is assumed that a satisfactory scheme of plant parameter identification is available. Information from the identifier is sent to a special purpose digital computer, into which has been programmed all the information necessary for the continual construction, search, and minimization of the cost subsurface appropriate to the plant, the compensator configuration, and to the model system. Optimum control parameter settings are then continually updated as the identifier and digital computer monitor the operation of the system.

Some further dividends are potentially available in the adaptive system outlined above. Many identification schemes require as much information as possible about the system's state variables; i.e., the system output and its successive derivatives. These variables are not often available, especially in higher-order systems. Identification then requires that some knowledge of some, if not all, of the state variables be gained by approximation techniques. As long as some of the higher derivatives are being approximated for use in identification, they might just as well simultaneously be used as feedback signals, enabling greater conformation of the actual system dynamics to those of the model. In the very unlikely case that all the state variables could be accurately approximated, the design procedure of this paper is not needed, nor for that matter is any other, since complete freedom would then exist for the placement of all system poles.

D. THE OPTIMUM SYSTEM AS A FUNCTION OF INITIAL CONDITIONS

The illustrative examples of Chapter III are all based upon an initial condition of displacement ($x_1(0) \neq 0$) only. This type of

disturbance was chosen because of its equivalence to a step input for a system with unity feedback in the outermost loop. This conforms to the wide use of the step as a test input in control system design.

It should be noted that the admission of non-zero values for the higher-order initial states will, for constrained systems, give rise to optimum free parameters that vary as functions of the direction of the initial condition vector in the initial condition vector space. This may be illustrated by referring to the cost associated with a second-order system, as a function of the initial conditions and the elements of the P matrix. This was given in equation (21) as

$$J = x_1^2(0)\, p_{11} + 2x_1(0)\, x_2(0)\, p_{12} + x_2^2(0)\, p_{22} \tag{63}$$

It is obvious from equation (63) that at least the shape of the total cost surface is a function of the initial conditions, and that the subsurfaces and their associated local minima generated by the actual system constraints will move about with changing initial conditions. It must be observed, however, that the minimum of the total cost surface is unalterably and securely tied to those coordinates specified by the coefficients of the model system. Equations (22) and (23) guarantee that the p_{ij}'s share a common minimum independent of the initial conditions, and similarly guarantee that any linear combination of the p_{ij}'s such as equation (63), also displays that same minimum. A completely free system, then, will always optimize at the model values. A system so constrained that the model poles are unattainable but relatively close to the attainable region will yield a narrow range of optimum solutions as initial conditions are changed.

It is not necessary to exhaust all points in the initial condition space to observe the migration of the optimum poles as a function of the initial conditions. Figure 17 shows the initial condition space for a second-order system. Point "1" is located on a unit circle centered at the origin. Point "2" shares a radial line with "1", and lies at a

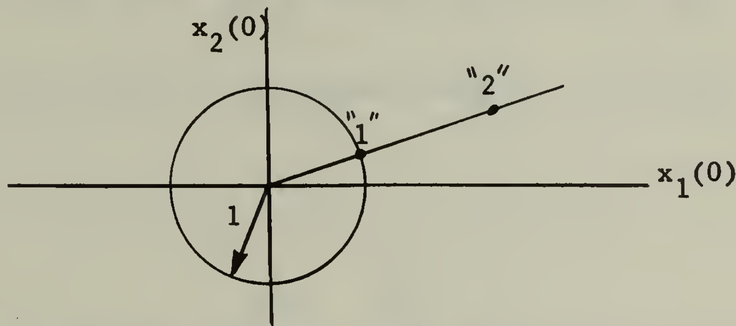
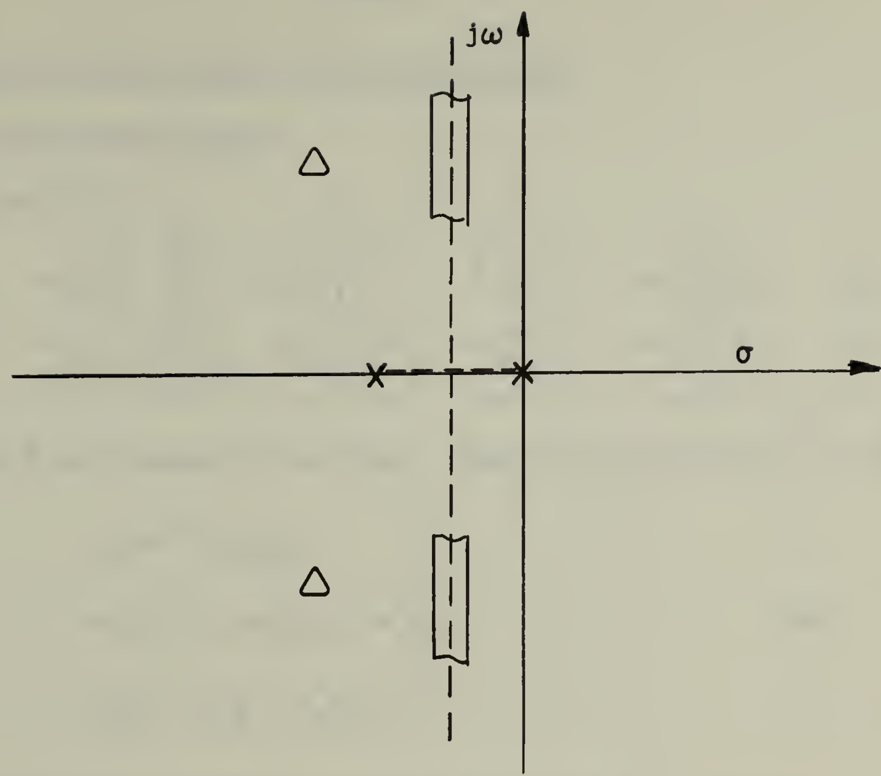


Fig. 17 Initial condition space, second-order system

radial distance d from the origin. It can be shown by simple substitution in equation (63) that the costs associated with points "1" and "2" differ only by the constant (d^2). This means that only the unit circle in the $x_1(0)$, $x_2(0)$ plane need be investigated. Points on this unit circle may be "mapped" onto the s-plane by determining the optimum pole locations for the constrained system corresponding to all points on that circle. The result will be a locus or region of optimum root locations for all possible initial conditions. The size and extent of the region, it is felt, will in large measure be dependent upon the proximity of the model poles to physically realizable pole locations. Such a region is shown in Figure 18, drawn for Example I. The end points of the optimum locus are not determined.



△ Model poles

--- Physically realizable root locus

Boxes enclose optimum portions of the locus

Fig. 18 Model poles and optimum pole locus for the conditions of Example I

It is felt that further investigation of these optimum regions presents a possible avenue for future research.

APPENDIX

I Solution for the P matrix from equation (13)

1) Second-order system

From equation (13),

$$\begin{bmatrix} -q_{11} & -q_{12} \\ -q_{21} & -q_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} + \begin{bmatrix} 0 & -a_1 \\ 1 & -a_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \qquad \text{(A1)}$$

where P and Q are symmetric matrices. Matrix multiplication yields

$$\begin{aligned} -q_{11} &= -2a_1p_{12} \\ -q_{12} &= p_{11} - a_2p_{12} - a_1p_{22} \\ -q_{22} &= 2p_{12} - 2a_2p_{22} \end{aligned} \qquad \text{(A2)}$$

Equations (A2) may be graphed as shown in Figure A1.

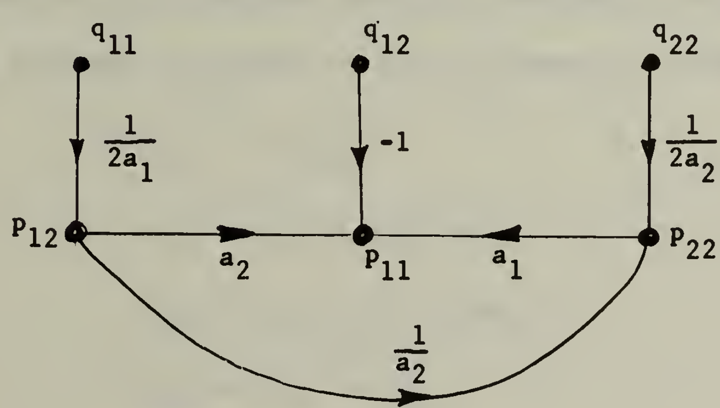


Fig. A1 Signal-flow graph for equations (A2)

The elements of the P matrix may be written directly from the graph, using Mason's gain formula.

$$p_{11} = -q_{12} + \frac{a_2}{2a_1} q_{11} + \frac{1}{2a_2} q_{11} + \frac{a_1}{2a_2} q_{22}$$

$$p_{12} = \frac{q_{11}}{2a_1} \qquad \text{(A3)}$$

$$p_{22} = \frac{q_{22}}{2a_2} + \frac{q_{11}}{2a_1 a_2}$$

2) Third-order system

From equation (13),

$$\begin{bmatrix} -q_{11} & -q_{12} & -q_{13} \\ -q_{21} & -q_{22} & -q_{23} \\ -q_{31} & -q_{32} & -q_{33} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -a_1 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \qquad \text{(A4)}$$

The nine scalar equations resulting from matrix multiplication in equation (A4) may be reduced to six by symmetry. They are

$$-q_{11} = -2a_1 p_{13}$$

$$-q_{12} = p_{11} - a_2 p_{13} - a_1 p_{23}$$

$$-q_{13} = p_{12} - a_3 p_{13} - a_1 p_{33}$$

$$-q_{22} = 2p_{12} - 2a_2 p_{33} \qquad \text{(A5)}$$

$$-q_{23} = p_{22} - a_3 p_{23} + p_{13} - a_2 p_{33}$$

$$-q_{33} = 2p_{23} - 2a_3 p_{33}$$

These equations are graphed in Figure A2.

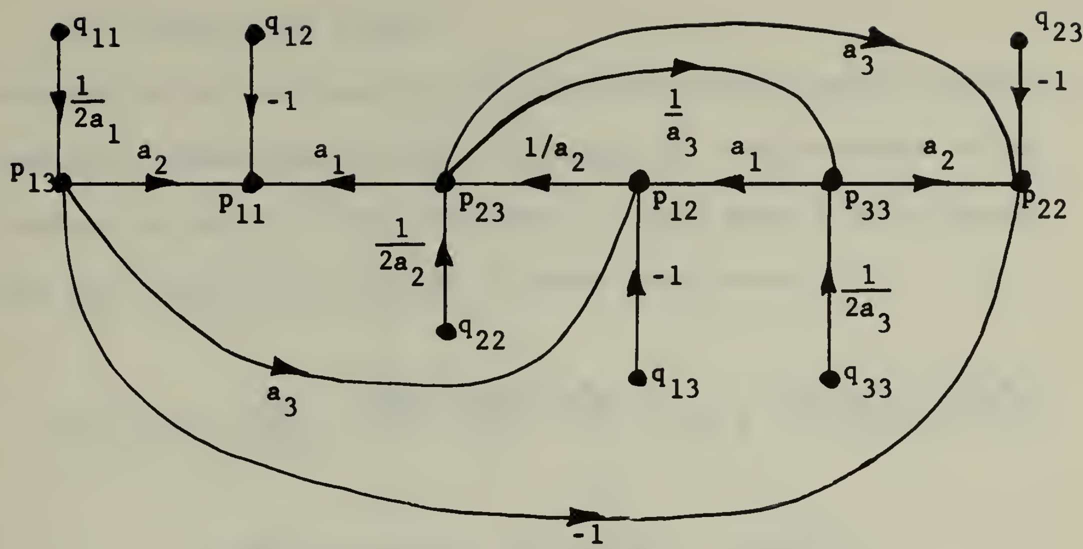


Fig. A2 Signal-flow graph for equations (A5)

Application of Mason's gain formula yields

$$
\begin{aligned}
p_{13} &= q_{11}/2a_1 \\
p_{11} &= q_{11}\left(\frac{a_2}{2a_1} + \frac{a_3^2}{2D}\right) - q_{12} + \frac{q_{22}a_1a_3}{2D} - \frac{q_{13}a_1a_3}{D} + \frac{q_{33}a_1^2}{2D} \\
p_{23} &= \frac{q_{11}a_3^2}{2a_1D} + \frac{q_{22}a_3}{2D} - \frac{q_{13}a_3}{D} + \frac{q_{33}a_1}{2D} \\
p_{12} &= -\frac{q_{13}a_2a_3}{D} + \frac{q_{33}a_1a_2}{2D} + \frac{q_{11}a_2a_3^2}{2a_1D} + \frac{q_{22}a_1}{2D} \qquad \text{(A6)} \\
p_{33} &= \frac{q_{11}a_3}{2a_1D} + \frac{q_{22}}{2D} - \frac{q_{13}}{D} + \frac{q_{33}a_2}{2D} \\
p_{22} &= -q_{23} + q_{11}\left(\frac{a_2a_3 + a_3^3}{2a_1D} - \frac{1}{2a_1}\right) + \frac{q_{33}(a_2^2 + a_1a_3)}{2D} \\
&\quad - \frac{q_{13}(a_2 + a_3^2)}{D} + \frac{q_{22}(a_2 + a_3^2)}{2D} \\
D &= a_2a_3 - a_1
\end{aligned}
$$

3) Fourth-order system

Manipulation of equation (13) for the fourth-order system yields ten equations, whose graph is shown in Figure A3. The elements of the P matrix may again be written directly, using Mason's gain formula. Only p_{11} is given here because of space limitations.

$$p_{11} = q_{11}\left(\frac{a_2}{2a_1} + \frac{a_3^2 a_4 - a_2 a_3 - a_1 a_4}{2D}\right) - q_{12} + \frac{q_{13}(a_1 a_2 - a_1 a_3 a_4)}{D}$$

$$+ \frac{q_{22}(a_1 a_3 a_4 - a_1 a_2)}{2D} + \frac{q_{24}(-a_1^2 a_4)}{D} + \frac{q_{33}(a_1^2 a_4)}{2D}$$

$$+ \frac{q_{44} a_1^2 a_2}{2D} \qquad \text{(A7)}$$

$$D = a_2 a_3 a_4 - a_1 a_4^2 - a_2^2$$

4) Fifth-order system

From equation (13), using signal-flow graphing, (not shown)

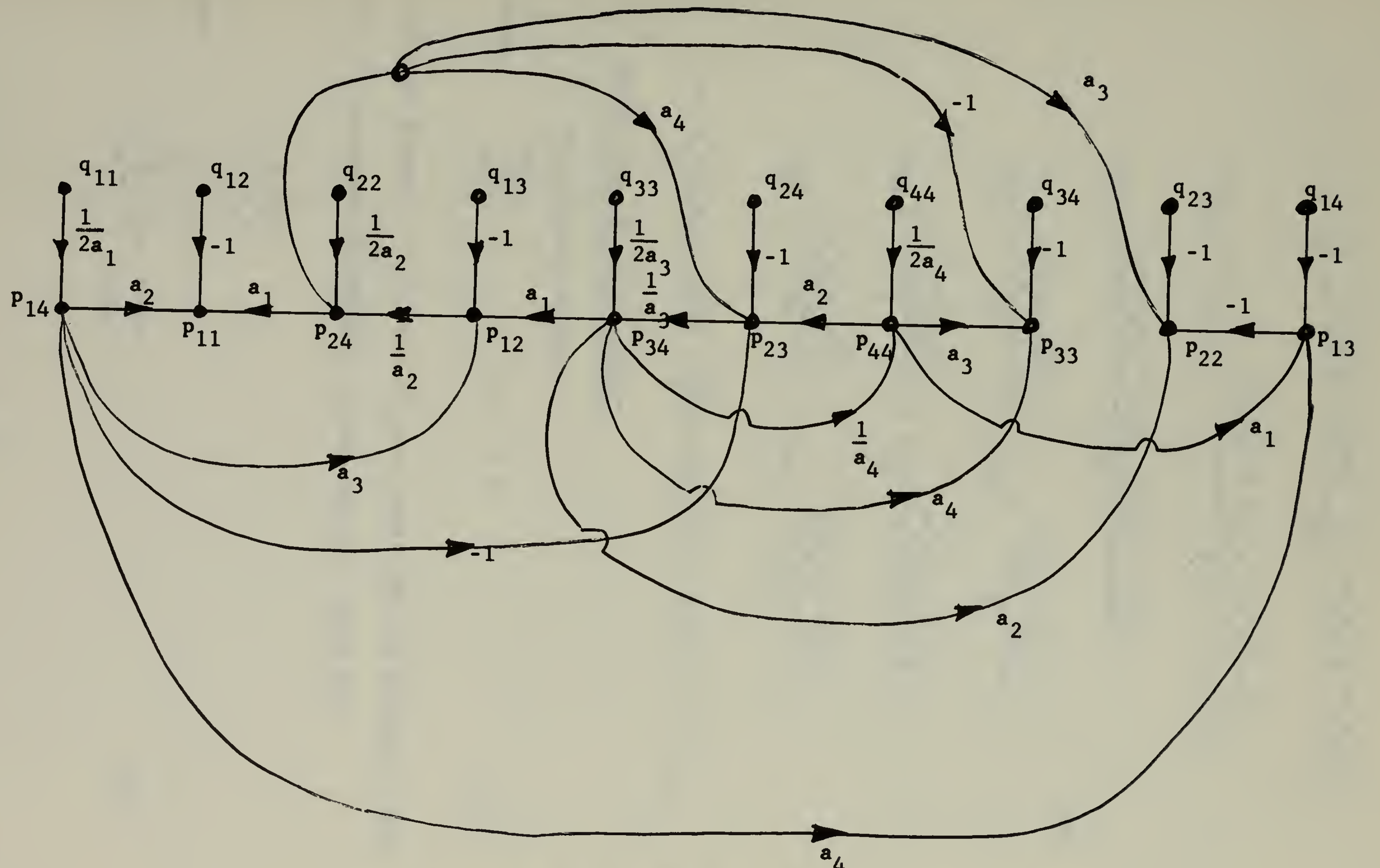


Fig. A3 Signal-flow graph for the Q, P equations of the fourth-order system

$$p_{11} = q_{11}\left(\frac{a_2}{2a_1} + \frac{a_3^2a_4a_5 - a_2a_3a_5^2 - a_3^2 + 2a_1a_3a_5 - a_1a_4a_5^2}{2D}\right)$$

$$+(-q_{12}) + q_{13}\left(\frac{a_1a_2a_5^2 - a_1a_3a_4a_5 + a_1a_3^2 - a_1^2a_5}{D}\right)$$

$$+q_{22}\left(\frac{a_1a_3a_4a_5 - a_1a_2a_5^2 - a_1a_3^2 + a_1^2a_5}{2D}\right)$$

$$+q_{33}\left(\frac{a_1^2a_4a_5 - a_1^2a_3}{2D}\right) + q_{24}\left(\frac{a_1^2a_3 - a_1^2a_4a_5}{D}\right) \qquad \text{(A8)}$$

$$+q_{35}\left(\frac{a_1^3 - a_1^2a_2a_5}{D}\right) + q_{15}\left(\frac{a_1^2a_4a_5 - a_1^2a_3}{D}\right)$$

$$+q_{44}\left(\frac{a_1^2a_2a_5 - a_1^3}{2D}\right) + q_{55}\left(\frac{a_1^2a_2a_3 - a_1^2a_4}{2D}\right)$$

$$D = a_2a_3a_4a_5 - a_1a_4^2a_5 - a_2^2a_5^2 - a_2a_3^2 - a_1^2 + 2a_1a_2a_5 + a_1a_3a_4$$

II <u>Derivation of the α's and λ</u>

1) Second-order system

From
$$\frac{\partial p_{ij}}{\partial a_i} = 0; \qquad i,\ j = 1 \to 2 \qquad \text{(A9)}$$

and Q as given in equation (18), six equations are obtained, only two of which are independent or non-trivial. They are

$$\lambda = \frac{1}{a_1^2}$$

and

$$\frac{\alpha_2}{2a_2^2} - \frac{\lambda}{2} + \frac{1}{a_1a_2^2} = 0 \qquad \text{(A10)}$$

[illegible]

[illegible]

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

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[illegible]

[illegible]

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$$[illegible]$$

from which

$$\alpha_2 = \frac{a_2^2 - 2a_1}{a_1^2} \tag{A11}$$

2) Third-order system

From

$$\frac{\partial p_{ij}}{\partial a_n} = 0 \; ; \qquad 1 \leq n \leq 3 \tag{A12}$$

i, j as in equations (A6)

and Q as given in equation (18), eighteen equations are obtained, only three of which are independent or non-trivial. They are

$$\lambda = \frac{1}{a_1^2}$$

$$a_1^2 a_2 \alpha_3 - a_2 a_3^2 + a_1^2 \alpha_2 + a_2^2 + 2a_1 a_3 = 0 \tag{A13}$$

$$a_1^3 \alpha_3 + a_1 a_3^2 + 2a_1 a_2 + a_1^2 a_3 \alpha_2 - a_2^2 a_3 = 0$$

The last two of equations (A13) may be solved for the α's to give

$$\alpha_2 = \frac{a_2^2 - 2a_1 a_3}{a_1^2}$$

$$\alpha_3 = \frac{a_3^2 - 2a_2}{a_1^2} \tag{A14}$$

3) Fourth-order system

Similar procedures applied to the fourth-order system yield four independent equations, whose solutions are

$$\lambda = \frac{1}{a_1^2}$$

$$\alpha = \frac{a_2^2 - 2a_1a_3}{a_1^2}$$

$$\alpha_3 = \frac{a_3^2 - 2a_2a_4 + 2a_1}{a_1^2} \qquad \text{(A15)}$$

$$\alpha_4 = \frac{a_4^2 - 2a_3}{a_1^2}$$

4) Fifth-order system

The fifth-order system yields the following solutions;

$$\lambda = \frac{1}{a_1^2}$$

$$\alpha_2 = \frac{a_2^2 - 2a_1a_3}{a_1^2}$$

$$\alpha_3 = \frac{a_3^2 - 2a_2a_4 + 2a_1a_5}{a_1^2} \qquad \text{(A16)}$$

$$\alpha_4 = \frac{a_4^2 - 2a_3a_5 + 2a_2}{a_1^2}$$

$$\alpha_5 = \frac{a_5^2 - 2a_4}{a_1^2}$$

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